

# Ribbon Operators and Hall-Littlewood Symmetric Functions

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**ABSTRACT.** Given a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ , let  $\lambda^{rc} = (\lambda_2 - 1, \lambda_3 - 1, \dots, \lambda_k - 1)$ . It is easily seen that the diagram  $\lambda/\lambda^{rc}$  is connected and has no  $2 \times 2$  subdiagrams which we shall refer to as a ribbon. To each ribbon  $R$ , we associate a symmetric function operator  $S^R$ . We may define the major index of a ribbon  $maj(R)$  to be the major index of any permutation that fits the ribbon. This paper is concerned with the operator  $H_{1^k}^q = \sum_R q^{maj(R)} S^R$  where the sum is over all  $2^{k-1}$  ribbons of size  $k$ . We show here that  $H_{1^k}^q$  has truly remarkable properties, in particular that it is a Rodriguez operator that adds a column to the Hall-Littlewood symmetric functions. We believe that some of the tools we introduce here to prove our results should also be of independent interest and may be useful to establish further symmetric function identities.

## 1. INTRODUCTION

The Schur functions indexed by a sequence of integers  $(p_1, p_2, \dots, p_k)$  can be defined by the Jacobi-Trudi identity  $s_{(p_1, p_2, \dots, p_k)} = \det |h_{p_i+i-j}|_{1 \leq i, j \leq k}$ . It is well known and easy to show that we have the relation

$$s_{(p_1, \dots, p_i, p_{i+1}, \dots, p_k)} = -s_{(p_1, \dots, p_{i+1}-1, p_i+1, \dots, p_k)}. \quad (1)$$

Let us recall that for a given symmetric function  $f$  it is customary to denote by  $f^\perp$  the operator that is dual to multiplication by  $f$  with respect to the Hall inner product. We shall make crucial use here of the Bernstein [5] operator

$$S_m = \sum_{k \geq 0} (-1)^k h_{m+k} e_k^\perp \quad (2)$$

Its action on the Schur basis may be easily computed with the formula  $S_m s_{(p_1, p_2, \dots, p_k)} = s_{(m, p_1, p_2, \dots, p_k)}$  and relation (1).

In particular we have the “Rodriguez” formula for Schur functions indexed by a partition  $\lambda$ ,

$$S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_k} 1 = s_\lambda \quad (3)$$

For a partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0)$  we set  $\ell(\lambda) = m$ . Partitions here are drawn by the French convention with the smallest part on the top.

A ribbon is a connected skew partition that contains no  $2 \times 2$  blocks. If  $\lambda$  is a partition of length  $k$ , we set  $\lambda^{rc} = (\lambda_2 - 1, \lambda_3 - 1, \dots, \lambda_k - 1)$  (the partition with the first row and column removed). It is easy to see that every ribbon partition will be  $\lambda/\lambda^{rc}$  for some partition  $\lambda$ .

Label the cells in a ribbon diagram with the numbers  $\{1, 2, \dots, k-1\}$  from left to right, top to bottom. This done, we let

$$D(R) = \{i \in [1, k-1] : i + 1^{\text{st}} \text{ cell of } R \text{ lies below the } i^{\text{th}} \text{ cell}\} \quad (4)$$

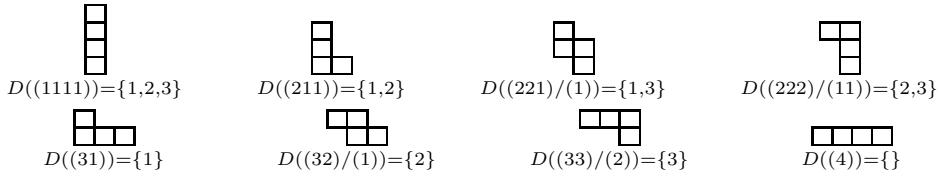
and refer to it as the descent set of  $R$ . The ribbons are therefore in one to one correspondence with the subsets of  $\{1, 2, \dots, k-1\}$ . If  $R$  is a ribbon of size  $k$ , we use  $\bar{R}$  to denote the ribbon whose descent set is  $\{1, 2, \dots, k-1\} - D(R)$ .

We also let the ‘major’ index of  $R$  be defined as

$$\text{maj}(R) = \sum_{i \in D(R)} i \quad (5)$$

and we set  $\text{comaj}(R) = \binom{k}{2} - \text{maj}(R)$  to be the complementary statistic. Clearly we have the relation  $\text{maj}(\bar{R}) = \text{comaj}(R)$ .

Below we have listed all of the ribbon partitions of size 4 with the corresponding descent set which is a subset of  $\{1, 2, 3\}$ .



We will use the symbol  $R$  to represent an arbitrary ribbon and the notation  $R \models k$  to indicate that  $R$  is a ribbon of size  $k$ .

For each ribbon  $R$  of size  $k$  we create an operator that raises the degree of the symmetric function by  $k$ . If  $R = \lambda/\lambda^{rc}$  then set

$$S^R = s_{\lambda^{rc}}^\perp \tilde{S}_{\lambda'_1} \tilde{S}_{\lambda'_2} \cdots \tilde{S}_{\lambda'_{\lambda_1}}. \quad (6)$$

where  $\lambda'_i$  is the length of the  $i^{\text{th}}$  column in the partition  $\lambda$ .

The action of this operator is very combinatorial in nature, we attach a ribbon on the left of the Schur function and reduce using the commutation relations  $\tilde{S}_a \tilde{S}_b = -\tilde{S}_{b-1} \tilde{S}_{a+1}$  and  $\tilde{S}_a \tilde{S}_{a+1} = 0$ , followed by the Littlewood-Richardson rule. We present one large example below.

$$S^{(432)/(21)}(s_{(33221)}) = -s_{(665)} - s_{(764)} - s_{(755)}$$

$$S \left( \begin{array}{c} \text{ribbon} \\ \text{ribbon} \end{array} \right) = \text{ribbon} = - \text{ribbon} = - \text{ribbon} - \text{ribbon} - \text{ribbon}$$

Define the symmetric functions  $H_\lambda[X; q] = \sum_\mu K_{\lambda\mu}(q) s_\mu[X]$  where  $K_{\lambda\mu}(q)$  is the Kostka-Folkes polynomial. There is an operator  $H_m^q$  that adds a row to this symmetric

function when  $m$  is larger than  $\lambda_1$  that is due to Jing [2]. In particular, it yields the ‘Rodriguez’ formula

$$H_{\lambda_1}^q H_{\lambda_2}^q \cdots H_{\lambda_k}^q 1 = H_{\lambda}[X; q]. \quad (7)$$

Our main result here is the construction of an operator  $H_{1^k}^q$  which adds a *column* to the partition indexing  $H_{\lambda}[X; q]$ . More precisely, we show here that the operator

$$H_{1^k}^q = \sum_{R \models k} q^{\text{comaj}(R)} S^R \quad (8)$$

has the following remarkable properties.

**Theorem 1.** *For all  $k \geq 0$ ,*

$$H_{m+1}^q H_{1^k}^q = H_{1^{k+1}}^q H_m^q. \quad (9)$$

As a result, we have for  $\ell = \ell(\lambda)$

$$H_{1^{\lambda_1}}^q H_{1^{\lambda_2}}^q \cdots H_{1^{\lambda_\ell}}^q 1 = H_{\lambda'}[X; q] \quad (10)$$

where  $\lambda'$  is the conjugate partition to  $\lambda$ .

The property that  $H_{1^k}^q$  adds a column to the Hall-Littlewood symmetric functions is a consequence of the commutation relation, since

$$H_{1^k}^q H_{\lambda}[X, q] = H_{\lambda_1+1}^q H_{\lambda_2+1}^q \cdots H_{\lambda_k+1}^q H_{1^0}^q(1) = H_{(\lambda_1+1, \lambda_2+1, \dots, \lambda_k+1)}[X; q]. \quad (11)$$

This result is the end product of a number of very interesting identities satisfied by ribbon operators. Our basic tool in establishing them is a truly remarkable new involution in the theory of symmetric functions. However we need to postpone the statement of these further results to the next section after we introduce some less familiar notation.

We should mention that (10) is a rather surprising extension to the general case of the classical identity

$$H_{1^k}[X; q] = \sum_{\sigma \in S_k} q^{\text{comaj}(\sigma)} s_{\lambda(\sigma)}[X] \quad (12)$$

where the sum is over the symmetric group  $S_k$  and  $\lambda(\sigma)$  denotes the shape of the standard tableaux corresponding to  $\sigma$  under Robinson-Schensted correspondence. In fact, by grouping terms according to descent sets we derive from (12) that

$$H_{1^k}[X; q] = \sum_{R \models k} q^{\text{comaj}(R)} s_R[X]. \quad (13)$$

**Example 2.** We see that for  $k = 3$ , formula (13) reduces

$$H_{(1^3)}[X; q] = \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} + q \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + q^2 \begin{smallmatrix} \square & \square \\ & \square \end{smallmatrix} + q^3 \begin{smallmatrix} \square & \square & \square \end{smallmatrix} \quad (14)$$

To compute the Hall-Littlewood symmetric function  $H_{(2^3)}[X; q]$  we act on the symmetric function  $H_{(1^3)}[X; q] = s_{(1^3)} + (q + q^2)s_{(21)} + q^3s_{(3)}$  with each ribbon of size 3.

$$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} H_{(1^3)}[X; q] = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + (q + q^2) \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + q^3 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \quad (15)$$

$$\begin{aligned} q \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} H_{(1^3)}[X; q] &= q \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + (q^2 + q^3) \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + q^4 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \\ &= (q^2 + q^3) \left( \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \right) + q^4 \left( \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \right) \end{aligned} \quad (16)$$

$$\begin{aligned} q^2 \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} H_{(1^3)}[X; q] &= q^2 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + (q^3 + q^4) \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + q^5 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \\ &= -q^2 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + q^5 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \end{aligned} \quad (17)$$

$$\begin{aligned} q^3 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} H_{(1^3)}[X; q] &= q^3 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + (q^4 + q^5) \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + q^6 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \\ &= q^6 \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix} \end{aligned} \quad (18)$$

We have therefore computed that  $H_{(2^3)}[X; q] = s_{(2^3)} + (q + q^2)s_{(321)} + q^3s_{(33)} + q^3s_{(4,1,1)} + (q^2 + q^3 + q^4)s_{(4,2)} + (q^4 + q^5)s_{(51)} + q^6s_{(6)}$ .

## 2. HATS AND RIBBONS

Let  $\Lambda$  represent the space of symmetric functions on an arbitrary number of variables considered as the polynomials over  $\mathbb{Q}$  in the power symmetric functions  $\{p_1, p_2, p_3, \dots\}$ . If  $\mathbf{X} = \{x_1, x_2, x_3, \dots\}$  is a set of variables then denote the symmetric functions in these variables by  $\Lambda^X$ . These two spaces are isomorphic and here we will often identify the two.

We will make use of a mix of plethystic notation and notation made standard by Macdonald in [3]. Plethystic notation is a device for expressing the substitution of the monomials of one expression in a symmetric function. Say that  $E$  is formal series in a set of variables  $x_1, x_2, \dots$  with possible special parameters  $q$  and  $t$  which should be thought of as unknown elements of  $\mathbb{Q}$ . For  $k \geq 1$ , define  $p_k[E]$  to be  $E$  with  $x_i$  replaced by  $x_i^k$  and  $q$  and  $t$  replaced by  $q^k$  and  $t^k$  respectively. For a symmetric function  $P$ ,  $P[E]$  will represent the the formal series found by expanding  $P$  in terms of the power

symmetric functions and then substituting  $p_k[E]$  for  $p_k$ . More precisely, if the power sum expansion of the symmetric function  $P$  is given by

$$P = \sum_{\lambda} c_{\lambda} p_{\lambda} \quad (19)$$

then  $P[E]$  is given by the formula

$$P[E] = \sum_{\lambda} c_{\lambda} p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]. \quad (20)$$

To evaluate a symmetric function in a set of variables  $\{x_1, x_2, x_3, \dots\}$ , set  $X = x_1 + x_2 + x_3 + \cdots$  and then we have that  $p_k[X] = x_1^k + x_2^k + x_3^k + \cdots$  and  $P[X]$  represents the symmetric function  $P$  evaluated at the  $x_i$ 's. For this exposition we will use capital letters  $X$  and  $Y$  to represent sums of infinite sets of variables,  $x_i$ 's and  $y_i$ 's respectively.

There is a well known scalar product on  $\Lambda$  defined by setting

$$\langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda\mu} z_{\lambda} = \delta_{\lambda\mu} \prod_{i \geq 1} i^{n_i(\lambda)} n_i(\lambda)! \quad (21)$$

where  $\delta_{xy}$  is the Kronecker delta and we have used the notation  $n_i(\lambda)$  to represent the number of parts of size  $i$  in  $\lambda$ .

For any symmetric function  $f$ , we denote the operation of skewing by  $f$  by  $f^{\perp}$  to represent the operation dual to multiplication by  $f$  with respect to this scalar product. More precisely,  $\langle f^{\perp}g, h \rangle = \langle g, fh \rangle$  and  $f^{\perp}g = \sum_{\lambda} s_{\lambda} \langle s_{\lambda} f, g \rangle$ .

For the Schur functions and the Hall-Littlewood symmetric functions, the vertex operators that add a row to the indexing symmetric function are well known. Define the two symmetric function operators and their generating functions  $S(z)$  and  $H(z)$  by

$$S_m P[X] = P \left[ X - \frac{1}{z} \right] \Omega[zX] \Big|_{z^m} = S(z) P[X] \Big|_{z^m} \quad (22)$$

$$H_m^q P[X] = P \left[ X - \frac{1-q}{z} \right] \Omega[zX] \Big|_{z^m} = H(z) P[X] \Big|_{z^m} \quad (23)$$

where  $\Omega[X] = \sum_{n \geq 0} h_n[X]$ . For a partition  $\lambda$ ,  $S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_{\ell}} 1 = s_{\lambda}[X]$  and  $H_{\lambda_1}^q H_{\lambda_2}^q \cdots H_{\lambda_{\ell}}^q 1 = H_{\lambda}[X; q] := \sum_{\mu \vdash |\lambda|} K_{\lambda\mu}(q) s_{\mu}[X]$  where  $K_{\lambda\mu}(q)$  are the Kostka-Foulkes coefficients. The operator  $S_m$  is due to Bernstein [5], and the operator  $H_m^q$  is due to Jing [2] although the notation and presentation here follows more closely [1].

Let  $P[X]$  be an arbitrary symmetric function in the set of  $X$  variables and let  $V$  be an operator (an element of  $\text{Hom}(\Lambda^X, \Lambda^X)$ ). Define  $\bar{V}$  by its action on  $P[X]$  by the following formula

$$\bar{V} P[X] = (V^Y P[X - Y]) \Big|_{Y=X} \quad (24)$$

The  $Y$  in the  $V^Y$  is there to emphasize that  $V$  is acting in the ‘dummy’  $Y$  set of variables only. After that operation is complete we set the  $Y$  variables equal to the  $X$  variables.

The hat operation also appears in a study of operators that add rows and columns to the standard bases of the symmetric functions [4]. The notation used here makes it a useful tool for deriving identities. The following proposition shows an important property of the hat operation, that it is an involution.

**Proposition 3.** *Let  $V$  be an element of  $\text{Hom}(\Lambda, \Lambda)$ .  $\overline{\overline{V}} = V$ .*

*Proof.* Let  $P[X]$  be an arbitrary symmetric function

$$\begin{aligned}\overline{\overline{V}}P[X] &= \overline{V}^Y P[X - Y] \Big|_{Y=X} \\ &= V^Z P[X - (Y - Z)] \Big|_{Z=Y} \Big|_{Y=X} \\ &= V^Z P[X - Y + Z] \Big|_{Y=X} \Big|_{Z=X} \\ &= V^Z P[Z] \Big|_{Z=X} = VP[X]\end{aligned}\tag{25}$$

◊

If  $R \models k$  and  $R^+ \models k+1$  such that  $D(R) = D(R^+)$  (that is,  $R^+$  is  $R$  with a cell to the right) then it follows directly from the definition

$$S^R \tilde{S}_1 = S^{R^+}.\tag{26}$$

A direct computation shows the following astonishing relation with the hat involution which provides a method for adding a cell below all others in the ribbon.

**Theorem 4.** *If  $R \models k$  and  $R_+ \models k+1$  such that  $D(R) \cup \{k\} = D(R_+)$ , then*

$$\overline{\overline{S^R}} \tilde{S}_1 = S^{R_+}\tag{27}$$

Before proceeding with the proof of this theorem, we remark that equations (26) and (27) imply that the following recursive definition is equivalent to the definition of  $H_{1^k}^q$  given in equation (8).

Let  $H_{1^0}^q = \tilde{S}_0$ ,  $H_{1^1}^q = \tilde{S}_1$ , and set

$$H_{1^k}^q = q^{k-1} H_{1^{k-1}}^q \tilde{S}_1 + \overline{\overline{H_{1^{k-1}}^q}} \tilde{S}_1\tag{28}$$

We will also need the following two lemmas to prove Theorem 4.

**Lemma 5.** For any operator  $V$ ,  $\overline{\overline{V}S_m} = \sum_{j \geq 0} (-1)^{m-j} h_j V \tilde{S}_{m-j}$

*Proof.* Let  $V$  be any operator.

$$\begin{aligned} \overline{\overline{V}S_m} &= V^W P[X - (Y - W) + 1/z] \Omega[z(Y - W)] \Big|_{W=Y} \Big|_{z^m} \Big|_{Y=X} \\ &= \Omega[zX] V^W P[W + 1/z] \Omega[-zW] \Big|_{W=X} \Big|_{z^m} \end{aligned} \quad (29)$$

Set  $\tilde{S}(z)P[X] = P[X + \frac{1}{z}] \Omega[-zX]$ , then  $\tilde{S}(z)P[X] \Big|_{z^\ell} = (-1)^\ell \tilde{S}_\ell P[X]$ . Take the coefficient of  $z^m$  to yield the identity.  $\diamond$

**Lemma 6.**  $\overline{\overline{s_\lambda^\perp S_{-m}}} = (s_{(m,\lambda)})^\perp$ .

*Proof.* The operator  $\overline{s_\lambda^\perp}$  has the action

$$\overline{s_\lambda^\perp} s_\mu[X] = s_\lambda^{\perp Y} s_\mu[X - Y] \Big|_{Y=X} = \sum_{\gamma} (-1)^{|\gamma|} s_{\mu/\gamma}[X] s_{\gamma'/\lambda}[X] = \begin{cases} (-1)^{|\lambda|} & \text{if } \lambda = \mu' \\ 0 & \text{otherwise} \end{cases} \quad (30)$$

Take the coefficient of  $s_\mu[Y]$  in the following equation.

$$\begin{aligned} \overline{\overline{s_\lambda^\perp S_{-m}}}^X \Omega[XY] &= \overline{s_\lambda^\perp}^U S_{-m}^U \Omega[XY] \Omega[-YU] \Big|_{U=X} \\ &= \Omega[XY] \overline{s_\lambda^\perp}^U S_{-m}^U \Omega \left[ -Y \left( U - \frac{1}{z} \right) \right] \Omega[zU] \Big|_{U=X} \\ &= \Omega[XY] \overline{s_\lambda^\perp}^X \Omega[-(Y - z)X] \Omega \left[ \frac{Y}{z} \right] \Big|_{z=-m} \\ &= \Omega[XY] s_\lambda[Y - z] \Omega \left[ \frac{Y}{z} \right] \Big|_{z=-m} \\ &= \Omega[XY] s_{(m,\lambda)}[Y] \end{aligned} \quad (31)$$

On the left one has  $\overline{\overline{s_\lambda^\perp S_{-m}}} s_\mu[X]$  and on the right  $s_{\mu/(m,\lambda)}[X]$ .  $\diamond$

*Proof.* (of Theorem 4) Say that  $S^R = s_{\lambda^{rc}}^\perp \tilde{S}_{\lambda'}$  for some partition  $\lambda$  and for brevity we have used the notation  $\tilde{S}_{\lambda'} := \tilde{S}_{\lambda'_1} \tilde{S}_{\lambda'_2} \cdots \tilde{S}_{\lambda'_m}$  with  $\lambda_1 = m$ .

$$\begin{aligned} \overline{\overline{S^R} S_1} &= \sum_{i \geq 0} (-1)^{1-i} h_i s_{\lambda^{rc}}^\perp \tilde{S}_{\lambda'} \tilde{S}_{1-i} \\ &= \sum_{i \geq 0} (-1)^{1+i+m} h_i s_{\lambda^{rc}}^\perp \tilde{S}_{1-i-m} \tilde{S}_{(m,\lambda)'} \\ &= \overline{\overline{s_{\lambda^{rc}}^\perp S_{1-m}}} \tilde{S}_{(m,\lambda)'} \end{aligned} \quad (32)$$

By the previous lemma this operator reduces to  $s_{(m-1,\lambda^{rc})}^\perp \tilde{S}_{(m,\lambda)'}$ , which is exactly  $S^{R+}$ .  $\diamond$

(26) and (27) provide a method for building all of the ribbon operators recursively. It follows in our next theorem that the hat involution sends ribbon operators to ribbon operators and permutes them in a very natural and non-trivial manner.

Say that two operators are  $k$ -level equal and write  $U \simeq_k V$  if  $V(s_\lambda) = U(s_\lambda)$  for all  $\ell(\lambda) \leq k$ .

**Theorem 7.** *Let  $R$  be a ribbon of size  $k$ , then*

$$\omega \overline{S^R} \omega \simeq_k S^{\overline{R}}. \quad (33)$$

We make the following general remark about operators and their hats before proceeding with the proof.

**Lemma 8.**  *$U \simeq_k V$  implies that  $\omega \overline{U} \omega \simeq_k \omega \overline{V} \omega$*

*Proof.* If  $\ell(\gamma) \leq k$  and  $U(s_\lambda) = V(s_\lambda)$  for all  $\lambda$  such that  $\ell(\lambda) \leq k$  then

$$\omega \overline{U} \omega(s_\gamma) = \sum_{\lambda \subseteq \gamma} (-1)^{|\lambda|} \omega(U(s_\lambda) s_{\gamma' \setminus \lambda'}) = \sum_{\lambda \subseteq \gamma} (-1)^{|\lambda|} \omega(V(s_\lambda) s_{\gamma' \setminus \lambda'}) = \omega \overline{V} \omega(s_\gamma) \quad (34)$$

◊

*Proof.* (of Theorem 7) The proof proceeds by induction on  $k$ . The result is true for  $R \models 1$  since  $\omega \overline{\tilde{S}_1} \omega(s_{(n)}) = s_{(n+1)} = \tilde{S}_1(s_{(n)})$ .

Let  $R^+$  be a ribbon of size  $k+1$ . Now either  $k \in D(R^+)$  and  $D(R^+) = D(R) \cup \{k\}$  for some ribbon  $R$  of size  $k$ , or  $k \notin D(R^+)$  and  $D(R^+) = D(R)$  for some ribbon  $R \models k$ .

In the first case we have that  $\omega \overline{S^{R^+}} \omega = \omega \overline{S^R} \tilde{S}_1 \omega = \omega \overline{S^R} \omega \tilde{S}_1$ . Note that if  $\ell(\lambda) = k+1$  and  $\tilde{S}_1(s_\lambda) = \pm s_\mu$  then  $\ell(\mu) = \ell(\lambda) - 1$ . Since  $\omega \overline{S^R} \omega$  is  $k$ -level equal to  $S^{\overline{R}}$  then we have  $\omega \overline{S^R} \omega \tilde{S}_1 \simeq_{k+1} S^{\overline{R}} \tilde{S}_1 = S^{R^+}$ .

In the second case, the same reasoning implies  $S^{R^+} = S^R \tilde{S}_1 \simeq_{k+1} \omega \overline{S^R} \omega \tilde{S}_1 = \omega \overline{S^R} S_1 \omega$ . By Theorem 4 and Lemma 8, this implies  $\omega \overline{S^{R^+}} \omega \simeq_{k+1} \overline{\overline{S^R} S_1} = S^{R^+}$ . ◊

### 3. PROOF OF THEOREM 1

**Lemma 9.** *For  $k \geq 2$*

$$q^{k-1} H_{1^k}^q \tilde{S}_2 = \overline{H_{1^k}^q S_2} \quad (35)$$

*Proof.* The left hand side of this equation may be expanded using  $\tilde{S}_1 \tilde{S}_2 = 0$ , equation (28) and Lemma 5, we have

$$\begin{aligned} q^{k-1} H_{1^k}^q \tilde{S}_2 &= q^{k-1} \left( q^{k-1} H_{1^{k-1}}^q \tilde{S}_1 + \overline{H_{1^{k-1}}^q S_1} \right) \tilde{S}_2 \\ &= q^{k-1} \overline{H_{1^{k-1}}^q S_1} \tilde{S}_2 \end{aligned} \quad (36)$$

The right hand side of the Lemma may also be expanded using the same relations. It follows that

$$\begin{aligned}
\overline{\overline{H_{1^k}^q} S_2} &= \sum_{j \geq 0} (-1)^{j+2} h_j H_{1^k}^q \tilde{S}_{2-j} \\
&= \sum_{j \geq 0} (-1)^{j+2} h_j \left( q^{k-1} H_{1^{k-1}}^q \tilde{S}_1 + \overline{\overline{H_{1^{k-1}}^q} S_1} \right) \tilde{S}_{2-j} \\
&= q^{k-1} \sum_{j \geq 0} (-1)^{j+2} h_j H_{1^{k-1}}^q \tilde{S}_1 \tilde{S}_{2-j} - \overline{\overline{H_{1^{k-1}}^q} S_1 S_2} \\
&= q^{k-1} \overline{\overline{H_{1^{k-1}}^q} S_1} \tilde{S}_2
\end{aligned} \tag{37}$$

hence (36) and (37) are equal.  $\diamond$

*Proof.* (of Theorem 1) Verify the following two identities by a direct computation.

$$H(u) \tilde{S}(z) P[X] = (q - u/z) \tilde{S}(z) H(u) P[X] \tag{38}$$

$$H(u) \Omega[zX] P[X] = \frac{1 - z/u}{1 - qz/u} \Omega[zX] H(u) P[X]. \tag{39}$$

Assume by induction for  $\ell < k$ ,  $H_{m+1}^q H_{1^\ell}^q = H_{1^{\ell+1}}^q H_m^q$ . Then in particular  $H(u) H_{1^{k-1}}^q = u H_{1^k}^q H(u)$ . By equation (29), we have

$$\begin{aligned}
H_{m+1}^q \overline{\overline{H_{1^{k-1}}^q} S_1} &= H(u) \Omega[zX] H_{1^{k-1}}^q \tilde{S}(z) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= \frac{1 - z/u}{1 - qz/u} \Omega[zX] u H_{1^k}^q (q - u/z) \tilde{S}(z) H(u) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= (u - u^2/z) \Omega[zX] H_{1^k}^q \tilde{S}(z) H(u) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= \overline{\overline{H_{1^k}^q} S_1} H_m^q - \overline{\overline{H_{1^k}^q} S_2} H_{m-1}^q
\end{aligned} \tag{40}$$

Using the same relations, we compute

$$\begin{aligned}
H_{m+1}^q H_{1^{k-1}}^q \tilde{S}_1 &= H(u) H_{1^{k-1}}^q \tilde{S}(z) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= u H_{1^k}^q (q - u/z) \tilde{S}(z) H(u) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= (qu - u^2/z) H_{1^k}^q \tilde{S}(z) H(u) \Big|_{z^1} \Big|_{u^{m+1}} \\
&= q H_{1^k}^q \tilde{S}_1 H_m^q + H_{1^k}^q \tilde{S}_2 H_{m-1}^q
\end{aligned} \tag{41}$$

Now using the recursive definition in equation (28) for  $H_{1^k}^q$  we have

$$H_{m+1}^q H_{1^k}^q = \overline{\overline{H_{1^k}^q} S_1} H_m^q + q^k H_{1^k}^q \tilde{S}_1 H_m^q - \overline{\overline{H_{1^k}^q} S_2} H_{m-1}^q + q^{k-1} H_{1^k}^q \tilde{S}_2 H_{m-1}^q \tag{42}$$

By Lemma 9 and equation (28), the right hand side reduces to  $H_{1^{k+1}}^q H_m^q$ .  $\diamond$

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